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Heatline and massline visualization of laminar natural convection boundary layers near a vertical wall

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Abstract

The development of the similarity version of the heatfunctions for the laminar natural convection near a vertical wall, isothermal or under constant heat flux, is presented for the first time in this work. From the beginning, the similarity formulation of the problem is based on a given form for the streamfunction, the streamlines not being usually presented. In this work, they are presented and analyzed. The paths followed by mass and heat are given by the streamlines and heatlines, which give us well bordered and non-crossed corridors where mass and heat are flowing. When the heatfunction is properly made dimensionless, its dimensionless values are closely related with the Nusselt number. In an attempt to present results that qualitatively apply over a wide range of Prandtl numbers, the used similarity variables are somewhat different from the ones usual in natural convection.

1. Introduction

The heatlines and masslines are the best way to visualize the heat and mass transfer occurring in a two-dimensional moving medium. The heatline concept was first introduced by Kimura and Bejan [1] and Bejan [2], and its mass counterpart, the massline concept, by Trevisan and Bejan [3]. The use of such concepts for visualization purposes is growing. The recent book by Bejan [4] reviews the pre-1995 literature on the heatlines and masslines, and uses widely such concepts to aid the visualization of some physical situations. However, in spite of its usefulness, the heatlines and masslines are not widely used by the numerical heat transfer community. The recent work by Costa [5] is an attempt to unify the streamline, heatline and massline concepts, in such a way that they can be easily introduced in the usual numerical heat transfer codes.

For a single component medium, the massfunction refers to the global mass, being thus the streamfunction, and the masslines are the streamlines. The constant values of the heatfunction and the streamfunction, the heatlines and streamlines, represent well bordered and non-crossed corridors, where heat and mass are flowing, the first due to the combined effects of convection and diffusion, and the later due to convection only (there is no diffusion in a single component medium). In unsteady situations, the heatline and massline concepts could be also applied, thus providing an instantaneous heat and mass flow visualization, as reported by Aggarwal and Manhapra [6]. When such tools are used to visualize the heat and mass transfer occurring in boundary layers, we can obtain a more complete picture of the involved transport phenomena as well as the true conditions where the boundary layer hypothesis are applied.

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Nomenclature

c_p	constant pressure specific heat
f	auxiliary similarity flow function
F	auxiliary function for the heatfunction
g	gravitational acceleration
g_i	(i = 1, 2) functions of the Prandtl number
Gr	Grashof number
h	convection heat transfer coefficient
H	heatfunction
k	thermal conductivity
L	a given level in the wall, $L \leq Y$
Nu	Nusselt number
Pr	Prandtl number
$\dot{q}''$	wall heat flux
$\dot{Q}$	heat flow at the wall
Ra	Rayleigh number
T	temperature
u, v	Cartesian velocity components
x, y	Cartesian co-ordinates
Y	highest level in the wall

Superscript

*	modified natural convection parameters

Subscripts

c	lowest value of temperature
cond	conduction
conv	convection
L	associated with length L
max	maximum value
min	minimum value
ref	reference value
T	referring to thermal effects
y	referring to level y
0	at the wall
*	dimensionless variable
∞	at the fluid reservoir, away from the wall

The boundary layer problem can be solved by the similarity method, the corresponding formulation being considerably simpler than the one expressed by the original partial differential equations. The heatfunction formulation can also be expressed in a similarity version, which uses some features of the original similarity problems' formulation. The heatfunction can be obtained in this way in closed forms (analytical forms), corresponding to different wall heating/cooling conditions. The situations of isothermal wall and constant heat flux at the wall (hot or cold wall) will be considered in this work. However, the most expressive form of the heatfunction is the one that takes as reference the lowest temperature value in the boundary layer [4]. This aspect precludes the finding of the closed form of the heatfunction corresponding to the constant heat flux cold wall situation. In order to obtain a formulation and results that apply over a wide range of Prandtl numbers, some of the used parameters and variables are different from the usual ones when analyzing the natural convection problem with the similarity methods.

This work is the natural convection counterpart of the one presented by Morega and Bejan [7] for forced convection boundary layers, and by Morega and Bejan [8] for forced convection boundary layers in fluid saturated porous media.

2. Mathematical modeling

The natural convection near the y-oriented wall of Fig. 1 is described by the mass conservation equation, the x and y momentum equations joined together in the y momentum boundary layer equation [4] and the thermal energy conservation equation, which reads, respectively,

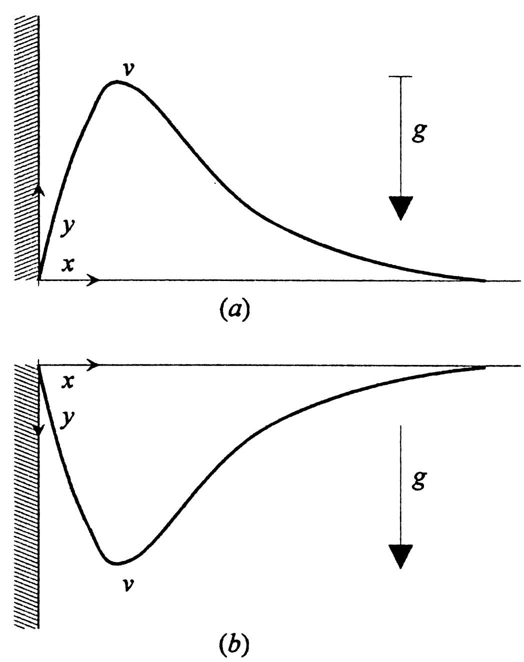

Fig. 1. Geometry for the situations analyzed: (a) hot wall; and (b) cold wall.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = v\frac{\partial^2 v}{\partial x^2}(\pm)g\beta(T - T_\infty), \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial x^2}. \tag{3}$$

This system of equations applies to the hot wall situation presented in Fig. 1a with the upper plus sign in Eq. (2), and to the cold wall situation presented in Fig. 1b with the lower minus sign in Eq. (2), a convention that will be used throughout this work.

It is assumed that the fluid is a constant property medium, the only exception being the density present in the buoyancy term, which has been made temperature dependent through the Boussinesq approach. The pressure gradient term equals the hydrostatic pressure gradient, the remaining buoyancy term being only $(\pm)g\beta(T - T_\infty)$. The boundary conditions for Eq. (2) are $u = v = 0$ at $x = 0$, and $v \to 0$ as $x \to +\infty$. For the thermal energy conservation equation (3), we always have that $T \to T_\infty$ as $x \to +\infty$. The boundary condition for Eq. (3) at $x = 0$ remains unspecified in order to consider further different heating/cooling wall conditions.

The heatfunction H is defined through its first-order derivatives as [5]

$$\frac{\partial H}{\partial y} = \rho u c_p(T - T_c) - k\frac{\partial T}{\partial x}, \tag{4a}$$

$$-\frac{\partial H}{\partial x} = \rho v c_p(T - T_c), \tag{4b}$$

where T_c is the *lowest* temperature in the boundary layer. The term $-k(\partial T/\partial y)$ is not present in Eq. (4b), as the boundary layer approach neglects the longitudinal y diffusion when compared with the longitudinal convection. This does not represent any limitation from the general heatfunction formulation [5], as the term $-k(\partial T/\partial y)$ can be considered and only taken as zero at the end of the development.

Evaluating and making the crossed derivatives of the heatfunction equal to 1 obtains the boundary layer energy conservation equation (3) identically, assuming implicitly that H is a C^2 continuous function. For the streamfunction, there is no additional consideration needed, as the similarity formulation of the problem usually starts considering a given form of the streamfunction.

3. Natural convection near an isothermal wall

3.1. Formulation

The boundary condition for the energy equation (3) at the wall is simply $T(0, y) = T_0$. If $T_0 > T_\infty$, we have the hot wall situation of Fig. 1a (plus sign in Eq. (2)), and if $T_0 < T_\infty$, we have the cold wall situation of Fig. 1b (minus sign in Eq. (2)).

The similarity formulation of the problem starts with the dimensionless similarity variable

$$\eta = \left(\frac{x}{y}\right)\left(\frac{Gr_y}{4}\right)^{1/4} \times g_1(Pr), \tag{5}$$

and the streamfunction

$$\psi = 4v\left(\frac{Gr_y}{4}\right)^{1/4} f(\eta) \times g_2(Pr), \tag{6}$$

where $Gr_y = g\beta|T_0 - T_\infty|y^3/v^2$ is the y-based Grashof number and $f(\eta)$ is a dimensionless auxiliary function of $O(1)$. It is usual to consider $g_1 = g_2 = 1$, essentially following the work of Ostrach [9]. However, in order to (qualitatively) enlarge the scope of the present results for different Prandtl number fluids, the $g_1(Pr)$ and $g_2(Pr)$ functions are defined as

$$g_1(Pr) = \left[\frac{4Pr^2}{(4 + Pr)}\right]^{1/4} \tag{7a}$$

$$g_2(Pr) = \left(\frac{\sqrt{2}}{2}\right)Pr^{-1}g_1(Pr). \tag{7b}$$

It should be noted that $(Gr_y/4)^{1/4} \times g_1 = [Ra_y\, Pr/(4 + Pr)]^{1/4}$, and that $4v(Gr_y/4)^{1/4} \times g_2 = 2\sqrt{2}\alpha[Ra_y\, Pr/(4 + Pr)]^{1/4}$, where $Ra_y = g\beta|T_0 - T_\infty|y^3/v\alpha$ is the y-based Rayleigh number. Each time we are searching for the solution for a given Prandtl number, that is, in a particular calculation Pr is taken as a constant and $f = f[\eta(x, y)]$ only, with $f' = df/d\eta$. However, strictly speaking, $\eta = \eta(x, y, Pr)$. Thus, for a given situation, the fluid flow solution is unique, and the ψ, v and u fields, and consideration of different forms of g_1 and g_2 leads to different values for the similarity function f.

The velocity components are evaluated as

$$u = \frac{\partial \psi}{\partial y} = \left(\frac{v}{y}\right)\left(\frac{Gr_y}{4}\right)^{1/4}(3f - \eta f') \times g_2 \tag{8a}$$

$$v = \frac{-\partial \psi}{\partial x} = -4\left(\frac{v}{y}\right)\left(\frac{Gr_y}{4}\right)^{1/2} f' \times g_1 g_2. \tag{8b}$$

The temperature difference is made dimensionless as

$$\theta[\eta(x, y, Pr)] = \frac{T(x, y) - T_\infty}{|T_0 - T_\infty|}, \tag{9}$$

given that, in any case, $\theta \to 0$ as $\eta \to +\infty$. At $\eta = 0$, we have that $\theta = 1$ for the hot wall situation and $\theta = -1$ for the cold wall situation. The denominator of Eq. (9) should be interpreted as $|T_0 - T_\infty| = T_{max} - T_{min}$.

The similarity version of the problem given by Eqs. (1)–(3) becomes

$$\theta'' = 3\,Pr\,f\,\theta' \times \frac{g_2}{g_1}; \quad \theta(0) = (\pm)1, \quad \theta(+\infty) = 0, \tag{10}$$

$$f''' = \left(3ff'' - 2f'^2\right) \times \frac{g_2}{g_1}(\pm)\theta \times \frac{1}{g_1^3 g_2}; \qquad f(0) = f'(0) = 0, \quad f'(+\infty) = 0, \tag{11}$$

which is solved using the shooting method.

For the heatfunction formulation the y variable is made dimensionless as $y_* = y/Y$, Y being the maximum wall length considered, the variable x is made dimensionless as $x_* = (x/Y)(Gr_Y/4)^{1/4} \times g_1$ and the heatfunction is made dimensionless as $H_* = H/(k|T_0 - T_\infty|)$. Recalling Eq. (5) one obtains $\eta(x_*, y_*) = x_* y_*^{-1/4}$.

Inserting the components of velocity given by Eqs. (8a) and (8b) in Eqs. (4a) and (4b), and noting that $(T - T_c) = |T_0 - T_\infty|\theta$ for the hot wall situation, and $(T - T_c) = |T_0 - T_\infty|(\theta - \theta_0)$ for the cold wall situation, one obtains

$$\frac{\partial H_*}{\partial y_*} = \left(\frac{Gr_Y}{4}\right)^{1/4} y_*^{-1/4}\left[Pr\,g_2\left(3f - \eta f'\right) \times \begin{pmatrix}\theta \\ \theta - \theta_0\end{pmatrix} - g_1\theta'\right], \tag{12a}$$

$$-\frac{\partial H}{\partial x_*} = -\left(\frac{Gr_Y}{4}\right)^{1/4} y_*^{1/2} 4 Pr\,g_2 f' \begin{pmatrix}\theta \\ \theta - \theta_0\end{pmatrix}. \tag{12b}$$

Assuming that H_* is of the form

$$H_*(x_*, y_*) = \left(\frac{Gr_Y}{4}\right)^{1/4} y_*^m F[\eta(x_*, y_*)], \tag{13}$$

and implicitly assuming that $H_*(0, 0) = 0$, one obtains that

$$\frac{\partial H_*}{\partial y_*} = \left(\frac{Gr_Y}{4}\right)^{1/4}\left(m y_*^{m-1} F - \frac{1}{4} y_*^{m-1}\eta F'\right), \tag{14a}$$

$$-\frac{\partial H_*}{\partial x_*} = -\left(\frac{Gr_Y}{4}\right)^{1/4} y_*^{m-1/4} F'. \tag{14b}$$

From Eqs. (12b) and (14b) one obtains an expression for F', which enters in Eq. (14a) that, conjugated with Eq. (12a), gives us the following expression for function F

$$F = \frac{1}{m} y_*^{-m+3/4}\left[3\,Pr\,g_2 f \begin{pmatrix}\theta \\ \theta - \theta_0\end{pmatrix} - g_1\theta'\right]. \tag{15}$$

Function F is independent of y_* if $m = 3/4$, and Eq. (13) becomes

$$H_*(x_*, y_*) = \frac{4}{3}\left(\frac{Gr_Y}{4}\right)^{1/4} y_*^{3/4}\left[3\,Pr\,g_2 f \begin{pmatrix}\theta \\ \theta - \theta_0\end{pmatrix} - g_1\theta'\right], \tag{16}$$

which is the *analytical* expression for the dimensionless heatfunction. The $H_*(x_*, y_*)$ field will be evaluated only after the f, θ and θ' fields are known, obtained after solving the system of equations (10) and (11).

3.2. Discussion

The local convection heat transfer coefficient at a given level y on the wall is obtained as

$$h_y = (\mp)\left(\frac{k}{y}\right)\left(\frac{Gr_y}{4}\right)^{1/4} g_1(\theta')_{\eta=0}. \tag{17}$$

The 0–L averaged convection heat transfer coefficient is obtained from Eq. (17) by integration, noting that $h_y \propto y^{-1/4}$, to give

$$h_{0-L} = (\mp)\left(\frac{4}{3}\right)\left(\frac{k}{L}\right)\left(\frac{Gr_L}{4}\right)^{1/4} g_1(\theta')_{\eta=0}. \tag{18}$$

The corresponding average Nusselt number, $Nu_{0-L} = (h_{0-L}L/k)$, is obtained as

$$Nu_{0-L} = (\mp)\left(\frac{4}{3}\right)\left(\frac{Gr_L}{4}\right)^{1/4} g_1(\theta')_{\eta=0} = (\mp)\left(\frac{4}{3}\right)\left(\frac{Gr_Y}{4}\right)^{1/4} L_*^{3/4} g_1(\theta')_{\eta=0}. \tag{19}$$

On the other hand, the dimensionless heatfunction on the level L in the wall is obtained from Eq. (16) as

$$H_*(0, L_*) = -(4/3)\left(\frac{Gr_Y}{4}\right)^{1/4} L_*^{3/4} g_1(\theta')_{\eta=0}, \tag{20}$$

the same result obtained in Eq. (19) for the 0–L averaged Nusselt number. It can be concluded that the dimensionless heatfunction, evaluated along the wall, gives us the average Nusselt number considered from the origin (0, 0) to the considered location (0, L) in the wall.

This is an expected result if we note that the heat flow exchanged between the wall and the fluid is given by

$$\dot{Q}_{0-L} = (\mp)(4/3)\left(\frac{Gr_Y}{4}\right)^{1/4} L_*^{3/4} g_1(\theta')_{\eta=0} k|T_0 - T_\infty|$$
$$= H(0, L).$$

This is easily interpreted returning to the physical meaning of the $H(0, L)$ value [5]: it is the value of the heat flowing, by unit length, L, between the points (0, 0) and (0, L) in the wall.

3.3. Illustration

The dimensionless streamlines and heatlines for the isothermal hot wall are presented in Fig. 2a and b, re-

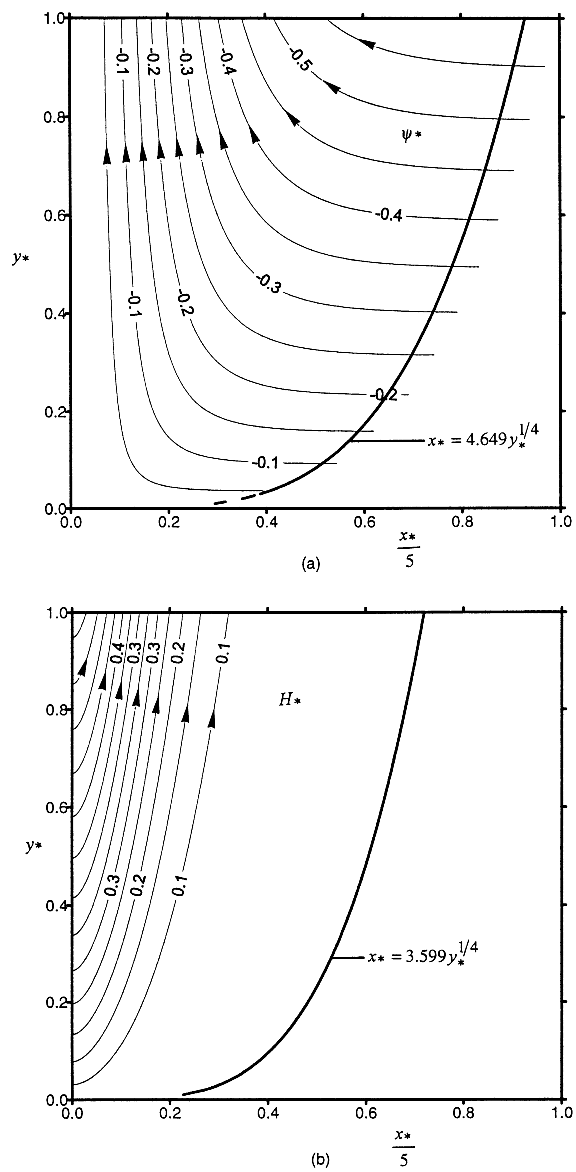

Fig. 2. Dimensionless results for an isothermal hot wall, for a $Pr = 0.73$ fluid: (a) streamlines; and (b) heatlines.

spectively, for a fluid medium with $Pr = 0.73$. The streamfunction is made dimensionless as $\psi_* = \psi / \lfloor 4\nu(\frac{Gr_Y}{4})^{1/4} \rfloor = y_*^{3/4} f g_2$. It was found from the similarity results that $(\eta)_{\delta_T} = 4.649$, that is, $x_* = 4.649 y_*^{1/4}$ at the exterior edge of the velocity boundary layer, a result also presented in Fig. 2a. At the exterior edge of the thermal boundary layer, $x_* = 3.599 y_*^{1/4}$, a result presented in Fig. 2b.

The first conclusion extracted from Fig. 2a and b is that mass and heat are flowing in counter-flow (in direction x) within the boundary layer, with a resulting upward motion of heat and mass. Another aspect is that there is an intense motion at the lower exterior edge of the velocity boundary layer, with the streamlines most dense for lower values of y_*. The main velocity changes occur effectively within the velocity boundary layer, as assumed by the boundary layer hypothesis. However, the hot wall presence is felt at the exterior edge of the velocity boundary layer in a considerable strength, with the mass flowing into the boundary layer.

As predicted by Eq. (17), $h_y \propto y^{-1/4}$, that is, the intensity of the heat transfer from the wall to the fluid is maximum for small y, and it decreases with $y^{-1/4}$ as y increases. This is shown in Fig. 2b, the heatlines at the wall being most sparse for higher y values. It is also found that the relevant heat transfer phenomena occurs inside the thermal boundary layer, as assumed by the boundary layer hypothesis.

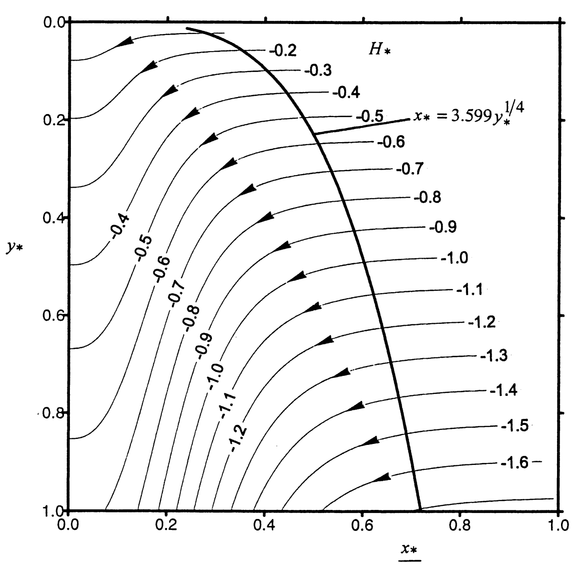

Fig. 3. Dimensionless heatlines for an isothermal cold wall, for a $Pr = 0.73$ fluid.

From Eq. (19) and Fig. 2b it can be concluded that the dimensionless heatfunction values at $\eta = 0$, for a given level L, can be interpreted as the average Nusselt number over the 0–L length. Thus, for $y_* = 1$ ($y = Y$), one obtains $H_*(0, 1) = Nu_{0-Y} = 0.676$.

For the isothermal cold wall situation, a figure for the streamlines can be obtained from Fig. 2a by a rotation of π around the x_*-axis, in such a way that the y_* axis points downward.

The corresponding heatlines are presented in Fig. 3. In this case, there is a parallel flow of mass and heat in the $-x$ direction, thus giving rise to a very different picture of the heatlines. These are considerably dense for lower y_* at the exterior edge of the thermal boundary layer. At $\eta = 0$, the picture of the heatlines is the same as when analyzing Fig. 2b, now with the y_* axis pointing downward. In this situation, there is an intense mass flow crossing the exterior edge of the velocity boundary layer, as for the hot wall situation, and also an intense heat flow (enthalpy flow) entering the thermal boundary layer through its exterior edge. In this sense, the cold wall presence is considerably felt, in both, the heat and fluid flow sense, even outside the thermal and velocity boundary layers, respectively. It can be seen from Fig. 3 that most of the heat entering the thermal boundary layer at a given level will reach the wall at a considerably lower level. It can be concluded from the streamline and heatline patterns that if the flow becomes turbulent for higher values of y_*, ($y_* \gg 1$), this will affect the heat and fluid flows at the $y_* < 1$ levels.

The presented results refer to a $Pr = 0.73$ fluid. However, as we are using the adequate variables in the similarity formulation, the qualitative behavior of the heatlines remains essentially the same even for very different Prandtl number fluids. More significant changes are expected on the streamlines behavior when changing the Prandtl number, essentially due to the considerably different picture of the velocity profiles for very different Prandtl number fluids.

4. Natural convection near wall with constant surface heat flux

4.1. Formulation

In this case, the boundary condition for the energy equation (3) at the wall is $k|\partial T/\partial x|_{x=0} = \dot{q}'' = \text{constant}$ (>0). If $(\partial T/\partial x)_{x=0} < 0$ (hot wall), we have $\dot{q}'' = -k(\partial T/\partial x)_{x=0}$, and $\dot{q}'' = k(\partial T/\partial x)_{x=0}$, otherwise. The reference work for this problem is that of Sparrow and Gregg [10].

The similarity formulation of the problem starts by considering the dimensionless similarity variable

$$\eta = \left(\frac{x}{y}\right)\left(\frac{Gr_y^*}{5}\right)^{1/5} g_1(Pr), \tag{21}$$

and the streamfunction

$$\psi = 5\nu\left(\frac{Gr_y^*}{5}\right)^{1/5} f(\eta) \times g_2(Pr), \tag{22}$$

where $Gr_y^* = g\beta\dot{q}''y^4/(k\nu^2)$ is the modified y-based Grashof number and $f(\eta)$ is, once again, a dimensionless auxiliary function of O (1). Analogously to what was made for the isothermal wall situation, to qualitatively enlarge the scope of the obtained results for different Prandtl number fluids, $g_1(Pr)$ and $g_2(Pr)$ functions are defined as

$$g_1(Pr) = \left\lfloor \frac{5\,Pr^2}{(7+Pr)} \right\rfloor^{1/5} \tag{23a}$$

$$g_2(Pr) = \left(\frac{\sqrt{15 \times 0.9 \times 7}}{15}\right) Pr^{-1} g_1(Pr). \tag{23b}$$

In this case, $(Gr_y^*/5)^{1/5} \times g_1 = [Ra_y^* Pr/(7+Pr)]^{1/5}$, and $\nu(\frac{Gr_y^*}{5})^{1/5} \times g_2 = (\sqrt{15 \times 0.9 \times 7}/3)\alpha[Ra_y^* Pr(7+Pr)]^{1/5}$, where $Ra_y^* = g\beta\dot{q}''y^4/(k\nu\alpha)$ is the y-based modified Rayleigh number.

Once again, each time we are searching for the solution for a given Prandtl number fluid, that is, in a particular calculation Pr is taken as constant and $f = f(\eta)$ only. Likewise, for a given situation the fluid flow solution is unique, and thus, the fields ψ, v and u, the consideration of different forms for g_1 and g_2 leading to different values for the similarity function f.

The velocity components are evaluated as

$$u = \frac{\partial\psi}{\partial y} = \left(\frac{\nu}{y}\right)\left(\frac{Gr_y^*}{5}\right)^{1/5}(4f - \eta f') \times g_2 \tag{24a}$$

$$v = -\partial\psi/\partial x = -5(\nu/y)(Gr_y^*/5)^{2/5} f' \times g_1 g_2. \tag{24b}$$

In this case, the temperature is made dimensionless as $\theta(\eta) = [T(x,\ y) - T_\infty]/\Delta T_y$, with $\theta \to 0$ as $\eta \to +\infty$. The temperature difference ΔT_y ($\Delta T_y > 0$) is not constant along the boundary layer height, being dependent on y. Considering that $\dot{q}'' = (\pm)k[T(0,\ y) - T_\infty]/\delta_{T,\ y}$, and that $\delta_{T,\ y} = y(\frac{Gr_y^*}{5})^{-1/5}(1/g_1)$, it can be stated that $\Delta T_y = (\dot{q}''/k)y(\frac{Gr_y^*}{5})^{-1/5}(1/g_1)$. Thus, the dimensionless temperature is expressed as

$$\theta[\eta(x, y, Pr)] = [T(x, y) - T_\infty]\left(\frac{k}{\dot{q}''}\right)\left(\frac{1}{y}\right)\left(\frac{Gr_y^*}{5}\right)^{1/5} g_1, \tag{25}$$

and, once the θ field is known, the generic temperature difference $[T(x, y) - T_\infty]$ can be obtained as

$$[T(x, y) - T_\infty] = \theta(\eta)\left(\frac{\dot{q}''y}{k}\right)\left(\frac{Gr_y^*}{5}\right)^{-1/5}\left(\frac{1}{g_1}\right). \tag{26}$$

The similarity version of the wall boundary condition $k|(\partial T/\partial x)_{x=0}| = \dot{q}''(\dot{q}'' > 0)$ for the energy equation (3) is expressed as $(\theta')_{\eta=0} = (\mp)1$.

The similarity version of the problem now becomes

$$\theta'' = Pr(4f\theta' - f'\theta) \times \frac{g_2}{g_1}; \quad \theta'(0) = (\mp)1, \quad \theta(+\infty) = 0, \tag{27}$$

$$f''' = (4ff'' - 3f'^{\,2}) \times \frac{g_2}{g_1}(\pm)\theta \times \frac{1}{g_1^4 g_2}; \quad f(0) = f'(0) = 0, \quad f'(+\infty) = 0, \tag{28}$$

which is solved using once again the shooting method.

For the heatfunction formulation, the variable y is made dimensionless as $y_* = y/Y$, Y being once again the maximum wall length considered, the variable x is made dimensionless as $x_* = (x/Y)(\frac{Gr_Y^*}{5})^{1/5} \times g_1$ and the heatfunction is made dimensionless as $H_* = H/(\dot{q}''Y)$. The variable η is obtained to depend on x_* and y_* as $\eta(x_*, y_*) = x_* y_*^{-1/5}$.

In this case, we have $(T - T_c) = \Delta T_y\theta$ for the hot wall situation and that $(T - T_c) = \Delta T_y(\theta - \theta_0 y_*^{-1/5})$ for the cold wall situation, as the minimum temperature within the considered boundary layer is $T(0, Y)$ in this case. It is this additional dependence of the temperature difference $(T - T_c)$ on y_* that precludes the use of the same procedure for the hot and cold wall situations. The heatfunction for the cold wall situation cannot be obtained in a closed analytical form, as is the case for the hot wall situation, and it needs the use of the general formulation presented by Costa [5].

For the hot wall situation, inserting the components of velocity given by Eqs. (24a) and (24b) in Eqs. (4a) and (4b), one obtains

$$\frac{\partial H_*}{\partial y_*} = Pr\,\frac{g_2}{g_1}(4f - \eta f')\theta - \theta', \tag{29a}$$

$$-\frac{\partial H_*}{\partial x_*} = -5\,Pr\,\frac{g_2}{g_1} y_*^{4/5}\theta. \tag{29b}$$

Similarly to what was made for the isothermal wall situation, one obtains the *analytical* expression for the heatfunction for the present situation as

$$H_*(x_*, y_*) = y_*\left(4\,Pr\frac{g_2}{g_1}f\theta - \theta'\right), \tag{30}$$

the field $H_*(x_*, y_*)$ being evaluated only after the fields f, θ and θ' are known from the solution of Eqs. (27) and (28).

4.2. Discussion

As the maximum temperature difference is not constant along the co-ordinate y, the definition of any average temperature difference leads to a well defined average heat transfer coefficient [11]. It should always be verified that $h_{0-L}|\Delta T_{0-L}| = \dot{q}''$. In this work, use is made of the fact that the surface heat flux is constant, essentially through the dimensionless heatfunction used.

In the present case, the physical interpretation of the heatfunction along the wall is easily obtained starting from Eq. (4a) with $(\partial H/\partial y)_{x=0} = -k(\partial T/\partial x)_{x=0}$. As $\dot{q}'' = C^{te}$ we have that $\dot{q}'' = (\mp)k(\partial T/\partial x)_{x=0}$, that is,

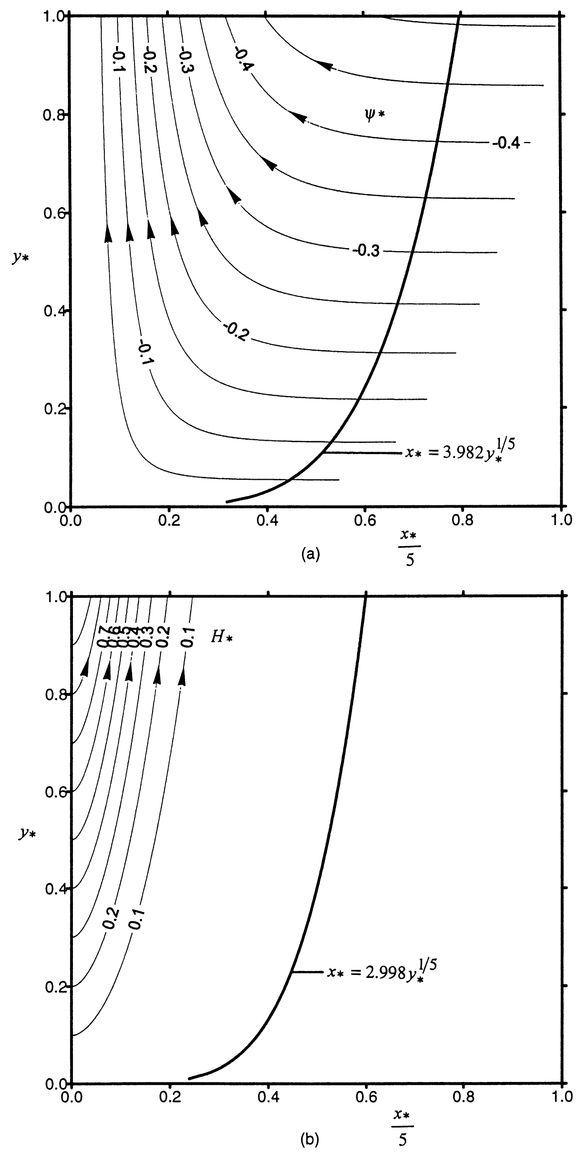

Fig. 4. Dimensionless results for a hot wall with constant surface heat flux, for a $Pr = 0.73$ fluid: (a) streamlines; and (b) heatlines.

$(\partial T/\partial x)_{x=0} = (\mp)(\dot{q}''/k)$. One obtains that

$$\left(\frac{\partial H}{\partial y}\right)_{x=0} = (\pm)\dot{q}''. \tag{31}$$

Integrating this equation along the wall, making $H(0, 0) = 0$, one obtains that

$$\int_0^L \left(\frac{\partial H}{\partial y}\right)_{x=0} \mathrm{d}y = \int_0^L (\pm)\dot{q}''\,\mathrm{d}y \Longrightarrow H(0, L) = (\pm)\dot{q}''L, \tag{32}$$

whose dimensionless version is

$$H_*(0, L_*) = (\mp)L_*, \tag{33}$$

where $L_* = L/Y$. This result is closely related with the 0–L average Nusselt number, interpreted in this case as the ratio

$$Nu_{0-L} = (\dot{Q}_{\text{conv}})_{0-L}/(\dot{Q}_{\text{cond}})_{0-Y} = (\dot{q}''L)/(\dot{q}''Y) = L_*. \tag{34}$$

4.3. Illustration

The dimensionless streamlines and heatlines corresponding to the hot wall situation, with a constant heat flux at the wall, for a $Pr = 0.73$ fluid, are presented in Fig. 4a and b, respectively. Now, the streamfunction is made dimensionless as $\psi_* = \psi/[5\nu(\frac{Gr^*_Y}{5})^{1/5}] = y_*^{4/5}fg_2$. In this case, $x_* = 3.982y_*^{1/5}$ at the exterior edge of the velocity boundary layer and $x_* = 2.998y_*^{1/4}$ at the exterior edge of the thermal boundary layer.

The main conclusions that can be extracted from these figures are essentially the same as when analyzing Fig. 2a and b, exceptions being the heatlines that are equally spaced at the wall, and $H_*(0, 1) = 1$. Such exceptions were naturally expected from the discussion made in Section 4.2.

The picture of the streamlines for the cold wall situation can be obtained from Fig. 4a by a rotation of π around the axis x_*, in such a way that the axis y_* points downward. The heatlines are not presented because, as referred above, they need to be obtained following a different procedure from the similarity-analytical one followed in this work.

5. Conclusions

The streamlines and heatlines are the most effective ways to visualize the paths followed by mass and heat flowing in two-dimensional problems without source terms in the energy equation. This is yet most relevant for boundary layer situations that admit a similarity formulation given that, in this case, an analytical (similarity) expression can be obtained for the heatfunction itself. This equally applies for forced or natural convection boundary layers. Additionally, through the streamlines and heatlines one can make a clearer critical analysis of the boundary layer hypothesis. However, it should be noted that we are evaluating the reasonability of a model through the results obtained with such a model. In this way, the obtained results will contain, in some extension, the implications of the model assumptions that we are evaluating.

The adequate explicit analytical similarity expression for the heatfunction can be obtained for isothermal hot or cold vertical walls, or for a hot vertical wall with constant heat flux at the surface. The heatfunction for the cold wall under the constant surface heat flux cannot be obtained in a closed form, but it can be obtained following the general heatfunction formulation.

When properly made dimensionless, the values of the heatfunction at the wall are closely related with the average Nusselt numbers corresponding to the wall–fluid or fluid–wall heat transfer. Using the adequate variables in the similarity formulation, the obtained pictures are valid, on a qualitative sense, for a wide range of Prandtl number fluids.

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